Tolerance Intervals for Poisson and Binomial Variables

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Quality control applications sometimes require one to obtain tolerance intervals for Poisson or binomial variables. Situations of this type are described and the construction of the desired interval is shown. This involves the combination of probability intervals and confidence intervals.

Introduction

THIS paper deals with the following two problems and their extensions.

Problem 1. The number of unscheduled shutdowns per year in a large population of systems follows a Poisson distribution with a constant shutdown rate from one system to the next and from one year to the next (see later discussion). The available data consist of a total of five system-years of operating experience and a total of twenty-four shutdowns. This information is to be used to obtain an upper bound on the number of shutdowns which one can claim with a high degree of confidence, such as 90 percent, will be exceeded by only a small percentage, such as five percent or fewer, of system-years in the sampled population. Equivalently, the given data are to be used to obtain an upper bound for which one can claim with a high degree of confidence that the number of shutdowns will equal or fall below for a high percentage, such as 95 percent or more, of all possible system-years. Such a bound is important in such practical problems as 1) determining the needed number of redundant systems and spare parts for each system, 2) deciding how large a staff of repair persons is required, and 3) guaranteeing system performance. This problem requires the determination of a so-called *upper tolerance bound*. In particular, an upper tolerance bound is a value calculated from the given data, which one can claim with a specified degree of confidence (100γ percent) equals or exceeds the values of a specified percentage ($100P$ percent) of the population values. Such an interval will, for brevity, be referred to as a (P, γ) upper tolerance bound.

Problem 2. A product is packed and shipped in cartons of forty-eight units. The units in a carton can be assumed to be a random selection from a production process. The process results in a constant unknown proportion p of defective units. In a random sample of 250 units from this (binomial) process, there were twenty defective units; the remaining 230 units were nondefective. These results are to be used to determine an upper tolerance bound on the number of defective units, which one can claim with 95 percent confidence will be exceeded by ten percent or fewer of the population of many cartons of forty-eight units. Equivalently, the given data are to be used to obtain an upper bound for which one can claim with 95 percent confidence that the number of defective units will be equal to or below for at least 90 percent of the cartons in the sampled population. The resulting (90, 95) tolerance bound on the number of defective units per carton is required to assess the degree of marketplace satisfaction with the product's performance.

The above two problems involve finding *one-sided upper tolerance bounds* in sampling from a Poisson distribution and from a binomial distribu-

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tion, respectively. In other instances, one might desire a *one-sided lower tolerance bound* or a *two-sided tolerance interval.* These are defined in a manner similar to that used to define an upper tolerance bound. In each case, two proportions, or percentages, are prespecified: 1) the proportion P of the population of interest which is to be covered by the calculated interval and 2) the level of confidence γ which one wishes to associate with the statement concerning this proportion. The data from a random sample are then used to obtain either a one-sided bound or two-sided bounds on the measured variable so that one may have 100γ percent confidence in the correctness of the claim that at least $100P$ percent of the population lies within the calculated bound(s).

The confidence statement concerning the population proportion P enters because the parameters of the underlying distribution, and therefore the proportion P, are generally not known. All that one has are the sample data. Such data introduce uncertainty in the estimates, and this uncertainty is taken into account by the confidence statement. When the underlying distribution is known exactly, the confidence interval converges to a probability interval. For example, for a normally distributed variable, with known mean μ and standard deviation σ, one knows with certainty that 95 percent of the population values are located in the probability interval $\mu \pm 1.96\sigma$. In practice, however, μ and σ are usually unknown and only their estimates, $\bar{x}$ and s, based upon a random sample of size n from the population, are available. As a result, one cannot, with certainty, determine an interval to contain a specified proportion of the population. Instead, one must construct a tolerance interval that one can claim, with a specified degree of confidence, covers a specified proportion of the population. For example, if the given sample size is $n = 10$, one can use the standard procedures for obtaining a (95, 90) tolerance interval for the normal distribution (see below) to claim with 90 percent confidence that at least 95 percent of the population values are located in the interval $\bar{x} \pm 3.018s$.

It is noted that sometimes practitioners construct a tolerance interval by assuming the unknown population parameter(s) to be equal to their sample estimates and then obtaining a probability interval. The resulting interval can be thought of as a tolerance interval with an associated confidence level of approximately 50 percent. Worse still, a confidence interval on a population parameter is sometimes incorrectly interpreted as a tolerance interval.

The best known tolerance intervals are those for a normal distribution, see Bowker (1947) and Wald and Wolfowitz (1947). Bowker and Lieberman (1959), Dixon and Massey (1969), Hahn (1970), Natrella (1966), and others provide expository discussions and tabulations of such intervals. Tolerance intervals can also be obtained based upon the ordered observations without making any assumptions about the underlying distribution. Such distribution-free tolerance bounds were originally developed by Wilks (1941) and are presented by Dixon and Massey (1969) and by Natrella (1966).

This paper shows how to construct tolerance intervals for Poisson and binomial variables; i.e., for the situations illustrated by Problems 1 and 2. The solutions are conceptually and operationally simple because both the Poisson and the binomial distributions involve only one unknown parameter (the mean occurrence rate λ for the Poisson distribution and the "success" probability p for the binomial distribution). As a result, construction of an upper tolerance bound to contain at least $100P$ percent of the population with 100γ percent confidence involves 1) obtaining an upper 100γ percent confidence bound for the unknown distribution parameter and 2) obtaining a $100P$ percent upper probability bound using the previously calculated upper confidence bound in place of the unknown parameter value. Details and illustrations are given in the balance of this paper. In particular, the next section deals with tolerance intervals (as well as probability intervals and confidence intervals) for a Poisson variable; the following section provides such intervals for binomial variables. Some extensions and related matters are discussed in the final section. The mathematical development of the results is given in a brief appendix.

Tolerance Intervals for a Poission Variable

Let the random variable X denote the number of shutdowns per system over a specified time period, such as a year, for a large population of a particular type of system. The distribution of X is Poisson if the shutdowns occur independently of one another and their mean occurrence rate is constant over time and from one system to the next. These assumptions would not hold and, therefore, the Poisson distribution would not be strictly applicable under either of the following circumstances. 1) Shutdowns are the result of a common environment such as unfavourable weather conditions, which has simultaneous impact on each of the systems. In this case shutdowns do not occur independently. Moreover, for independence to hold, the occurrence of a shutdown should not increase or decrease the chances of a subsequent shutdown on the same system. 2) The mean occurrence rate of shutdowns changes over the life of a system due, perhaps, to system wear-out and is thus not constant over time.

The Poisson distribution would appear to be applicable, for example, if shutdowns were attributable to purely external factors, such as mishandling during shipment or improper use, which affect each of the systems separately. In practice, one would want to examine the sample data to assess the validity of these assumptions. In particular, Hoaglin (1979) recently proposed a graphical procedure for assessing Poissoness and the classical chi-square test for this purpose [see Hahn and Shapiro (1967), ch. 8] provides a formal statistical test. Note also that the Poisson distribution differs from the binomial distribution in that rather than dealing with a dichotomy of system success and failure, numerous occurrences on the same system over a period of time are possible. This would be the case for a repairable system. For some situations, it would also be convenient to approximate the binomial distribution by the Poisson distribution, see Hahn and Shapiro (1967).

The Poisson probability function is

$$f(x, \lambda) = \frac{\lambda^x e^{-\lambda}}{x!} \qquad x = 0, 1, 2, \ldots \tag{1}$$

where x is a particular outcome of the random variable X and the parameter λ denotes the mean occurrence rate of X. We review below construction of a probability interval for a Poisson variable with known parameter λ and construction of a confidence interval to contain the unknown parameter λ, on the basis of sample data. These results are then used to obtain the desired Poisson distribution tolerance intervals. The discussion will be illustrated by Problem 1 of the Introduction.

Probability Intervals for a Poisson Variable

In constructing a probability interval, all distribution parameters are assumed to be known. Thus, for a Poission variable, the mean occurrence rate λ is assumed to be known. In particular, a lower $100P$ percent probability bound for a Poisson variable is the maximum number of occurrences N_L which will be equalled or exceeded with a probability of at least $100P$ percent, where P is frequently taken to be a number such as 0.90, 0.95, or 0.99. Equivalently, it is the maximum number of occurrences N_L such that the probability of falling below N_L is $100(1 - P)$ percent or less. Thus, for a Poisson variable the lower $100P$ percent probability bound is the value N_L that satisfies both equations (2) and (3).

$$G(N_L; \lambda) = \sum_{x=N_L}^{\infty} \frac{\lambda^x e^{-\lambda}}{x!} \geq P \tag{2}$$

$$G(N_L + 1; \lambda) = \sum_{x=N_L+1}^{\infty} \frac{\lambda^x e^{-\lambda}}{x!} < P \tag{3}$$

Or equivalently, it is one that satisfies both equations (4) and (5).

$$F(N_L - 1; \lambda) = \sum_{x=0}^{N_L-1} \frac{\lambda^x e^{-\lambda}}{x!} \leq 1 - P \tag{4}$$

$$F(N_L; \lambda) = \sum_{x=0}^{N_L} \frac{\lambda^x e^{-\lambda}}{x!} > 1 - P \tag{5}$$

Similarly, an upper $100P$ percent probability bound for a Poisson variable is the minimum number of occurrences N_U such that the probability of N_U or fewer occurrences taking place is at least $100P$ percent. Equivalently, it is the minimum number of occurrences N_U such that the probability of exceeding N_U is $100(1 - P)$ percent or less. Thus, for a Poisson variable the upper $100P$ percent probability bound is the value N_U that satisfies both (6) and (7).

$$F(N_U; \lambda) = \sum_{x=0}^{N_U} \frac{\lambda^x e^{-\lambda}}{x!} \geq P \tag{6}$$

$$F(N_U - 1; \lambda) = \sum_{x=0}^{N_U-1} \frac{\lambda^x e^{-\lambda}}{x!} < P \tag{7}$$

Or equivalently, it is one that satisfies both equations (8) and (9).

$$G(N_U + 1; \lambda) = \sum_{x=N_U+1}^{\infty} \frac{\lambda^x e^{-\lambda}}{x!} \leq 1 - P \tag{8}$$

$$G(N_U; \lambda) = \sum_{x=N_U}^{\infty} \frac{\lambda^x e^{-\lambda}}{x!} > 1 - P \tag{9}$$

A two-sided $100P$ percent probability interval involves the simultaneous specification of both a lower probability bound N_L and an upper probability bound N_U, such that the probability of the number of occurrences being at least N_L and at most N_U is at least $100P$ percent. Equivalently, the interval consists of a pair of values, N_L and N_U, such that the probability of the number of occurrences being less than N_L or more than N_U is $100(1 - P)$ percent or less. If one adds the symmetry requirement that the lower bound of the interval is also a one-sided lower $[100(1 + P)/2]$ percent probability bound and that the upper end of the interval is also an upper [100(1 + P)/2] percent probability bound, one then simply takes N_L as the one-sided lower [100(1 + P)/2] percent probability bound, and N_U as the one-sided upper [100(1 + P)/2] percent probability bound. For example, a two-sided 95 percent probability interval would be obtained by using the combination of a one-sided lower 97.5 percent probability bound and a one-sided upper 97.5 percent probability bound. As will be seen, a symmetric two-sided probability interval such as this is sometimes more conservative than would be the case if symmetry were not required.

The preceding probability bounds have been defined in terms of the probability of a random observation falling within the constructed interval. The bounds may equivalently be defined in terms of the proportion of the population located within that interval. For example, an upper 95 percent probability bound N_U is the minimum number of occurrences which is not exceeded by 95 percent of the members of the population.

The laborious calculations for obtaining the desired probability bounds suggested by the preceding development do not usually need to be performed directly. Instead, one can utilize available tabulations of summations of the Poisson probability function (i.e., its cumulative distribution function). Relevant tabulations include the following.

1. Defense Systems Department, General Electric Company (1962). Individual terms [see equation (1)] and cumulative terms [see equations (2) through (9)] are given for the following.
   λ = 0.00000010(0.00000001)0.00000015 (0.0000005)0.000015(0.0000001)0.00005 (0.000005)0.0005(0.00001)0.001(0.00005) 0.005(0.0001)0.01(0.0005)0.2(0.001)0.4 (0.005)0.5(0.01)1(0.05)2(0.1)5(0.5)10(1) 100(5)205
   In this tabulation, the cumulative terms are given in both the form $F(x; \lambda)$
   $$F(x; \lambda) = \sum_{k=0}^{x} \frac{e^{-\lambda}\lambda^k}{k!}$$
   and also in the form $G(x; \lambda)$.
   $$G(x; \lambda) = \sum_{k=x}^{\infty} \frac{e^{-\lambda}\lambda^k}{k!} = 1 - F(x-1; \lambda)$$
2. Molina (1949). Individual and cumulative terms are given for the following.
   λ = 0.001(0.001)0.01(0.01)0.3(0.1)15(1)100
   In this tabulation, the cumulative terms are given in the form $G(x; \lambda)$.
3. Pearson and Hartley (1966). Individual terms are given (in Table 39) for
   λ = 0.0005(0.0005)0.005(0.005)0.05(0.05)1.0 (0.1)5(0.25)10(0.5)20(1)60
   and cumulative terms are given in (Table 7) for λ = 0.1(0.1)15, i.e., for $\lambda = 0.1$ to $\lambda = 15$ in increments of 0.1. In this tabulation the cumulative terms are given in the form $F(x-1; \lambda)$.

In addition, Dodge and Romig (1944) give charts for the cumulative Poisson probabilities for values of λ from 0.1 to 30. These charts are also given by Hahn and Shapiro (1967). Finally, computer libraries can be consulted for routines for evaluating individual or cumulative Poisson distribution probabilities or such routines could be easily developed.

Example (Based on Problem 1)

Unscheduled system breakdowns follow a Poisson distribution with a known mean occurrence rate $\lambda = 4.8$. The construction of (a) a lower 95 percent probability interval, (b) an upper 95 percent probability interval (the interval actually required in Problem 1), and (c) a two-sided 95 percent probability interval will be shown.

a. Lower 95 percent probability bound. For λ = 4.8 one obtains $G(2; 4.8) = 0.952$ and $G(3; 4.8) = 0.857$ from Molina (1949). Thus, $N_L = 2$ satisfies both equations (2) and (3) and is the desired lower 95 percent probability bound; i.e., fewer than five percent of the systems in the sampled population will have fewer than two shutdowns in a year. This result is conservative since, due to the discrete nature of the Poisson distribution, only an integer number of shutdowns is possible. Thus, $N_L = 2$ is actually a lower 95.2 percent probability bound since the probability of two or more shutdowns is 0.952.

b. Upper 95 percent probability bound. For λ = 4.8, one obtains $G(9; 4.8) = 0.0558$ and $G(10; 4.8) = 0.0251$ from Molina (1949). Thus, $N_U = 9$ satisfies both equations (8) and (9) and is the desired 95 percent probability bound; i.e., fewer than five percent of the systems will have more than nine shutdowns in a year. Again this result is conservative; $N_U = 9$ is actually an upper 97.49 percent probability bound since the probability of nine or fewer shutdowns is 0.9749.

c. Two-sided 95 percent probability interval. Using the same approach as above, one obtains

1. a lower 97.5 percent probability bound of $N_L = 1$ and
2. an upper 97.5 percent probability bound of $N_U = 10$.

Thus, $N_L = 1$ to $N_U = 10$ is a two-sided symmetric 95 percent probability interval on the number of occurrences. (Note: A tighter two-sided 95 percent probability interval could be obtained if symmetry, i.e., requiring a probability of at most 2.5 percent of falling beyond each endpoint, were not required. In particular, (1, 9) is also a two-sided 95 percent probability interval; however, for the interval (1, 9), the probability of exceeding the upper bound of nine is greater than 2.5 percent).

Confidence Intervals for the Poisson Parameter λ

Assume again that X is a Poisson variable. However, assume this time that the value of the parameter λ is not known and needs to be estimated from the given data. For example, say that there have been twenty-four unscheduled shutdowns in five system-years of operation. (The number of systems upon which these results are based and the operating time of any individual system in the sample do not matter if the number of unscheduled shutdowns per system per year are independent Poisson variates, as is being assumed.) The given data may be used to obtain one-sided or two-sided 100γ percent confidence bound(s) to contain λ. Specifically, let λ_L and λ_U denote the lower and upper confidence bounds on λ, respectively.

A simple procedure for obtaining λ_L and λ_U is to refer to a table especially prepared for this purpose; for example, Table 40 of Pearson and Hartley (1966), which also appears in Hahn and Shapiro (1967), p. 160. This table provides one-sided upper and lower 100γ percent confidence bounds on λ for $\gamma = 0.999$, 0.995, 0.990, 0.975 and 0.950 based on $x = 0(1)30(5)50$ observed occurrences. Using both the lower and the upper confidence bounds together, one obtains symmetric two-sided confidence bounds for $\gamma = 0.998$, 0.990, 0.980, 0.950, and 0.900, respectively. Actually, this tabulation provides confidence bounds, λ'_L and λ'_U, in terms of the *total observed time period* T'; for example, five system-years. To obtain the confidence bounds λ_L and λ_U for a specified *unit time period* T, such as one system-year, the tabulated values λ'_L and λ'_U must be multiplied by the factor T/T', as will be illustrated in the example.

Tables of 100γ percentage points, $\chi^2(\nu; \gamma)$, of the chi-square distribution with parameters or degrees of freedom ν may also be used for obtaining confidence bound(s) on λ due to the relationship between the Poisson and chi-square distributions, see Pearson and Hartley (1966) p. 10. In particular, to find λ_L, based upon x Poisson occurrences (e.g., system shutdowns), one proceeds as follows.

1. Set $\nu = 2x$.
2. Use a tabulation of the percentage points of the chi-square distribution (see below) to find $\chi^2(\nu; 1 - \gamma)$.
3. Find the lower confidence bound λ'_L in terms of the observed time period as $\lambda'_L = (1/2)\chi^2(\nu; 1 - \gamma)$.
4. Then find the lower confidence bound in terms of the specified unit time period, $\lambda_L = \lambda'_L(T/T')$.

To find λ_U one proceeds as follows.

1. Set $\nu = 2(x + 1)$.
2. Use a tabulation of the percentage points of the chi-square distribution to find $\chi^2(\nu; \gamma)$.
3. Find the upper confidence bound λ'_U in terms of the observed time period as $\lambda'_U = (1/2)\chi^2(\nu; \gamma)$.
4. Then find the upper confidence bound in terms of the specified time period, $\lambda_U = \lambda'_U(T/T')$.

The 100γ percent points $\chi^2(\nu; \gamma)$ of the chi-square distribution are tabulated in numerous books on statistics such as Bowker and Lieberman (1959), Dixon and Massey (1969), Hahn and Shapiro (1967), Natrella (1966), and Pearson and Hartley (1966). For example, Table 8 of Pearson and Hartley (1966) provides tabulations for the following.

$\gamma = 0.005$, 0.010, 0.025, 0.050, 0.100, 0.250, 0.500, 0.750, 0.900, 0.950, 0.975, 0.990, 0.995, 0.999

$\nu = 1(1)30(10)100$

Example

Assume that the number of unscheduled system shutdowns over a specified time period are independent Poisson events with a common, but unknown, mean shutdown rate λ per system per year. In a random five system-year period, there has been a total of twenty-four such shutdowns. In this problem, the observed time period is $T' = 5$ system-years and the specified time period of interest is $T = 1$ system-year. The two-sided 90 percent, upper 90 percent, and lower 90 percent confidence bounds to contain λ are obtained as follows.

a. Two-sided (symmetric) 90 percent confidence interval. From Table 40 of Pearson and Hartley (1966) or page 16 of Hahn and Shapiro (1967), the two-sided 90 percent confidence interval to contain the mean shutdown rate λ' per five system-years is (16.55, 33.75) based upon $x = 24$. Thus, the desired two-sided 90 percent confidence interval to contain λ is $\lambda_L = 16.55(1/5) = 3.31$ to $\lambda_U = 33.75(1/5) = 6.75$.

b. Lower 90 percent confidence bound λ_L. The tabulations in Hahn and Shapiro (1967) and Pearson and Hartley (1966) cannot be used since they do not provide one-sided bounds for $\gamma = 0.90$. Instead, a tabulation of the percentage points of the chi-square distribution is used as follows.

1. Set $\nu = 2(24) = 48$.
2. Although $\chi^2(48; 0.10)$ is not tabulated in Table 8 of Pearson and Hartley (1966), $\chi^2(40; 0.10)$ is equal to 29.1 and $\chi^2(50; 0.10)$ is equal to 37.7; thus by linear interpolation one obtains $\chi^2(48; 0.10) = 36.0$.
3. The lower 90 percent confidence bound on λ' is $\lambda'_L = 36.0/2 = 18.0$.
4. The desired lower 90 percent confidence bound on λ is $\lambda_L = 18.0(1/5) = 3.6$.

c. Upper 90 percent confidence bound λ_U. A tabulation of the percentage points of the chi-square distribution is used as follows.

1. Set $\nu = 2(24 + 1) = 50$.
2. Obtain $\chi^2(50; 0.90) = 63.2$ from Table 8 of Pearson and Hartley (1966).
3. The upper 90 percent confidence bound on λ' is $\lambda'_U = 63.2/2 = 31.6$.
4. The desired upper 90 percent confidence bound on λ is $\lambda_U = 31.6(1/5) = 6.32$.

Tolerance Intervals for a Poisson Variable

Obtaining a tolerance interval for a Poisson variable involves a combination of the two preceding procedures. An interval similar to a probability interval in that it is to contain a specified proportion of the population values is desired. However, the parameter λ is now not known exactly and must be estimated from the sample data. One thus proceeds as follows in obtaining a (P, γ) tolerance interval.

1. Obtain the 100γ percent confidence interval to contain the unknown population parameter λ from the sample data. If a one-sided lower (or upper) tolerance bound is desired, find the one-sided 100γ percent lower (or upper) confidence bound on λ, i.e., λ_L (or λ_U). For a two-sided tolerance interval, find the two-sided 100γ percent confidence interval to contain λ.
2. Obtain a probability interval to contain at least $100P$ percent of the population using the confidence bound(s) λ_L and/or λ_U in place of the unknown parameter value λ.
   a. If a lower tolerance bound is required, obtain a lower $100P$ percent probability bound using the lower confidence bound λ_L in place of the unknown λ.
   b. If an upper tolerance bound is required, obtain an upper $100P$ percent probability bound using the upper confidence bound λ_U in place of the unknown λ.
   c. If a two-sided tolerance interval is required, use λ_L in place of λ in calculating the lower bound of the two-sided $100P$ percent probability interval, and use λ_U in place of λ in calculating the upper bound of the $100P$ percent probability interval.

Examples of Calculating Poisson Distribution Tolerance Intervals

Consider again Problem 1 of the Introduction. The number of unscheduled shutdowns per system per year are independent Poisson variates. The given data consist of twenty-four unscheduled system shutdowns over five system-years. An upper (95, 90) tolerance bound N_U is desired. Thus, N_U is the number of shutdowns per system-year which one can claim with 90 percent confidence will be exceeded by fewer than five percent of the large population of system-years. To illustrate the procedure, a lower one-sided tolerance bound and a two-sided tolerance interval are also calculated.

a. Lower tolerance bound. As seen above, the lower 90 percent confidence bound on λ, based on the given data, is $\lambda_L = 3.6$. A lower 95 percent probability bound on a Poisson variable with $\lambda = 3.6$ is required. From Molina (1949) one obtains $G(1; 3.6) = 0.972$ and $G(2; 3.6) = 0.874$. Thus, from (2) and (3), $N_L = 1$ is the desired lower tolerance bound; i.e., one can state with 90 percent confidence that at least 95 percent of the systems in the sampled population will have one or more unscheduled shutdowns in a given year. (This result is conservative due to the discrete nature of the Poisson distribution; it actually corresponds to at least 97.2 percent of the systems, rather than to at least 95 percent.) Note that a lower tolerance bound with a 90 percent associated confidence level is interpreted as follows. If many bounds of this type are calculated from different independent samples of data, the resulting claim that the stated population percentage exceeds the calculated lower tolerance bound will be correct in 90 percent of the cases. The calculated upper tolerance bound and two-sided tolerance interval are similarly interpreted.

b. Upper tolerance bound. As seen above, the upper 90 percent confidence bound on λ, based on the given data is $\lambda_U = 6.32$. An upper 95 percent probability bound on a Poisson variable with $\lambda = 6.32$ is required. By interpolating in Molina (1949) one obtains $G(11; 6.32) = 0.057$ and $G(12; 6.32) = 0.028$. Thus, from (8) and (9), $N_U = 11$ is the desired upper tolerance bound; i.e., one can state with 90 percent confidence that at least 95 percent of the systems in the sampled population will have eleven or fewer unscheduled shutdowns in a given year. (This result is also conservative; coincidentally, it again corresponds to 97.2 percent of the system.)

c. Two-sided tolerance interval. The lower and upper bounds of the two-sided 90 percent symmetric confidence interval on the Poisson parameter λ are, respectively, the one-sided lower 95 percent confidence bound $\lambda_L = 3.31$ and the one-sided upper 95 percent confidence bound $\lambda_U = 6.75$. To obtain the lower bound of the two-sided symmetric 95 percent tolerance interval, a lower 97.5 percent probability bound on a Poisson variable with $\lambda = 3.31$ is required. This turns out to be $N_L = 0$. To obtain the upper bound of the two-sided tolerance interval an upper 97.5 percent probability bound on a Poisson variable with $\lambda = 6.75$ is required. This turns out to be $N_U = 12$. Thus, (0, 12) is the desired two-sided symmetric tolerance interval; i.e., one can state with 90 percent confidence that at least 95 percent of the systems in the sampled population will have from 0 to 12 (inclusive) unscheduled shutdowns in a given system-year. This contrasts with the previously obtained probability interval (1, 10) which results if one simply uses the sample point estimate $\lambda = 4.8$ in place of the unknown parameter λ. The increased length of the interval is due to the uncertainty in one's knowledge of λ.

Tolerance Intervals for a Binomial Variable

Assume that a sample of n units is randomly selected from a large population with unknown proportion defective p. The total number of defectives X in the sample follows a binomial distribution with probability distribution function

$$f(x; n, p) = \frac{n!}{x!(n-x)!} p^x (1-p)^{n-x} \qquad (10)$$

where x is a particular value of the random variable X, $x = 0, 1, 2, \ldots, n$. The following description of the construction of a tolerance interval for a binomial variable is organized in a manner similar to the discussion of that for a Poisson variable given in the preceding section and will be illustrated by Problem 2 of the Introduction. However, the discussion here will be less detailed than before and only the construction of a one-sided upper tolerance bound will be considered. Finding a one-sided lower tolerance bound or a two-sided tolerance interval follows in the same manner as previously.

Probability Intervals for a Binomial Variable

An upper $100P$ percent probability bound for the number of defectives in a random sample of n units from a binomial population with a proportion defective p is the minimum number N_U such that the probability that N_U or fewer defectives are selected is at least $100P$ percent; equivalently, it is the minimum number N_U such that the probability of exceeding N_U defectives is $100(1 - P)$ percent or less. Thus, the upper tolerance bound is the value N_U that satisfies both equations (11) and (12).

$$F(N_U; p, n) = \sum_{x=0}^{N_U} \frac{n!}{x!(n-x)!} \cdot p^x(1-p)^{n-x} \geq P \qquad (11)$$

$$F(N_U - 1; p, n) = \sum_{x=0}^{N_U-1} \frac{n!}{x!(n-x)!} \cdot p^x(1-p)^{n-x} < P \qquad (12)$$

Or equivalently, it must satisfy both equations (13) and (14).

$$G(N_U + 1; p, n) = \sum_{x=N_U+1}^{n} \frac{n!}{x!(n-x)!} \cdot p^x(1-p)^{n-x} \leq 1 - P \qquad (13)$$

$$G(N_U; p, n) = \sum_{x=N_U}^{n} \frac{n!}{x!(n-x)!} \cdot p^x(1-p)^{n-x} > 1 - P \qquad (14)$$

Available tabulations of the binomial distribution function can be used to perform the evaluations required by equations (11) through (14). These include the following.

1. AMCP 706-109 (1972). Cumulative terms [see equations (11) through (14)] are given for the following.

   $x = 1(1)n$
   $n = 1(1)150$
   $p = 0.01(0.01)0.5$ and $0.001(0.001)0.010$

   In this tabulation, the cumulative terms are given in the following form.

$$G(x; p, n) = \sum_{c=x}^{n} \frac{n!}{c!(n-c)!} p^c(1-p)^{n-c} \qquad (15)$$

   For $p > 0.5$, one uses the tabulation in conjunction with the following relationship.

$F(x; p, n) = 1 - G(n - x; 1 - p, n)$

2. National Bureau of Standards (1950). Individual terms [see equation (10)] and cumulative terms of the form $G(x; p, n)$ are given for the following.
   $x = 0(1)n$
   $n = 2(1)49$
   $p = 0.01(0.01)0.50$
3. Romig (1953). Individual terms and cumulative terms are given for the following.
   $x = 0(1)n$
   $n = 50(5)100$
   $p = 0.01(0.01)0.50$
4. *Tables of Cumulative Binomial Probability Distribution* (1955). Cumulative terms are given for the following.
   $x = 0(1)n$
   $n = 1(1)50(2)100(10)200(50)1000$
   $p = 0.01(0.01)0.50$; 1/16, 1/12, 1/8, 1/6, 3/16, 5/16, 1/3, 3/8, 5/12, 7/16
   In this tabulation, the cumulative terms are given in the form $G(x; p, n)$.
5. Weintraub (1963). Cumulative terms of the form $G(x; p, n)$ are given for the following.
   $x = 0(1)n$
   $n = 1(1)100$
   $p = 0.0001(0.0001)0.0009(0.001)0.100$

For large n, the binomial distribution can be approximated by a normal distribution with mean $\mu = np$ and variance $\sigma^2 = np(1 - p)$. In particular

$$F(x; p, n) = \Phi[(x + (1/2) - \mu)/\sigma] = P \quad (16)$$

$$G(x; p, n) = 1 - \Phi[(x - (1/2) - \mu)/\sigma] = 1 - P \quad (17)$$

where $\Phi(z)$ is the probability of a standard normal variate falling below z. This is tabulated in most texts on statistics. (The factor 1/2, which is added to and subtracted from x in (16) and (17) respectively, is a continuity correction. It is used because x is being treated as a continuous variable for all possible values of X between $x - 1/2$ and $x + 1/2$.) The preceding approximation gives reasonable results if np and $n(1 - p)$ are both at least five. Its use is illustrated in the numerical example at the end of this section.

Example (Based on Problem 2)

A process produces a known proportion $p = 0.08$ defective units. The number of defective units X in a random sample of $n = 48$ units is a binomial variable. A 90 percent upper probability bound on the number of defective units in such a sample is desired. For $p = 0.08$ and $n = 48$ one obtains $G(6; 0.08, 48) = 0.18293$ and $G(7; 0.08, 48) = 0.08599$ from *Tables of Cumulative Binomial Probability Distribution* (1955). Thus, $N_U = 6$ satisfies both equations (13) and (14) and is the desired (conservative) 90 percent upper probability bound; in other words, assuming a constant process proportion defective of 0.08, fewer than ten percent of the large population of cartons of forty-eight units will contain more than six defective units.

Confidence Intervals for the Binomial Parameter *p*

Assume again that X is a binomial variable. However, now assume that the value of the parameter p is unknown and needs to be estimated from the given data. Specifically, assume that in a random sample of n units from a large population there has been a total of x defective units. This information can be used to obtain one-sided or two-sided 100γ percent confidence bounds on p. Some methods for obtaining such bounds are as follows.

1. Graphical method. Dixon and Massey (1969), Natrella (1966), and others give charts for obtaining one-sided 80 percent, 90 percent, 95 percent, and 99 percent confidence bounds to contain p for $n = 5(5)20,30,100,250,1000$ as a function of the observed x/n (on the abscissa). Combining upper and lower bounds together, these graphs provide two-sided 60 percent, 80 percent, 90 percent, and 98 percent confidence bounds, respectively. In Pearson and Hartley (1966), Table 41 provides graphs for two-sided 95 and 99 percent confidence bounds for $n = 8(1)12(4)24$, 30, 40, 60, 100, 200, 400, 1000. Visual interpolation may be used for intermediate values of n. These graphs do not provide the bounds very precisely; they may suffice, however, for many practical situations.

2. Tabular method. Natrella (1966) gives tabulations of both one-sided (Table A-23) and two-sided (Table A-24) 90, 95, and 99 percent confidence bounds to contain p for $n = 1(1)30$. The two-sided confidence bounds are not symmetric, but are minimum length bounds, see Crow (1956). This may lead to somewhat different bounds than the other methods given here.

3. Normal distribution approximation. For sufficiently large sample sizes, approximate confidence bounds to contain p may be obtained by the use of the normal distribution approximation to the binomial distribution. In particular, if both np and $n(1 - p)$ exceed five, a good approximate 100γ

percent one-sided confidence bound on p [see Dixon and Massey (1969)] is given by

$$\frac{n}{n+z_\gamma^2}\left\{\hat{p} \pm \frac{1}{2n} + \frac{z_\gamma^2}{2n} \pm z_\gamma\left[\frac{\left(\hat{p} \mp \frac{1}{2n}\right)\left(\hat{q} \pm \frac{1}{2n}\right)}{n} + \frac{z_\gamma^2}{4n^2}\right]^{1/2}\right\} \quad (18)$$

where $\hat{p} = x/n$, $\hat{q} = 1 - \hat{p}$ and z_γ denotes the 100γ percent point of the standard normal distribution. The upper signs are used for finding the upper confidence bound p_U and the lower signs for the lower confidence bound p_L. A coarser approximation, which can be used for large n, is given by the following.

$$\hat{p} \pm z_\gamma[\hat{p}(1-\hat{p})/n]^{1/2} \quad (19)$$

In both cases, the endpoints of one-sided lower and upper $100[1-(1-\gamma)/2]$ percent confidence bounds provide a two-sided 100γ percent confidence interval. See Dixon and Massey (1969) for further details.

4. Other methods. Some other methods for obtaining a confidence bound to contain p are
   a. use of a standard computer program such as the General Electric Information Service Company Mark III time-sharing routine CONDIF*** [see *Quick-Use Programs* (1972)];
   b. interpolation in a tabulation of the cumulative binomial distribution such as National Bureau of Standards (1950), Romig (1953), *Tables of Cumulative Binomial Probability Distribution* (1955), and Weintraub (1963) [see *Tables of Cumulative Binomial Probability Distribution* (1955) p. xxiv–xxxiii for further discussion];
   c. binomial probability paper [see Mosteller and Tukey (1949)];
   d. Poisson distribution approximation to the cumulative binomial distribution for small p [see Hahn and Shapiro (1967)]; and
   e. normal distribution approximation with arcsin transformation [see Hald (1952) for discussion of this and further methods].

Example

A random sample of 250 units from a process with an unknown proportion of defectives p contains a total of twenty defective units. Then, using the normal distribution approximation, an upper 95 percent confidence bound on p is

$$p_U = \frac{250}{250+(1.645)^2} \cdot \left\{0.08 + \frac{1}{500} + \frac{(1.645)^2}{500} + (1.645) \cdot \left[\frac{(0.08-1/500)(0.92+1/500)}{250} + \frac{(1.645)^2}{4(250)^2}\right]^{1/2}\right\}$$
$$= 0.115$$

using (18) with $z_{0.95} = 1.645$, $n = 250$, $\hat{p} = 20/250 = 0.08$ and $\hat{q} = 1 - 0.08 = 0.92$. Use of the coarser normal distribution approximation [equation (19)] gives the following upper 95 percent confidence bound.

$$p_U = 0.080 + 1.645[(0.08)(1-0.08)/250]^{1/2}$$
$$= 0.108$$

Tolerance Bounds for a Binomial Variable

Obtaining an upper (P, γ) tolerance bound for a binomial variable involves a combination of the two preceding procedures. In particular, one proceeds as follows.

1. Find the upper 100γ percent confidence bound p_U on p from the sample data.
2. Obtain a probability interval to contain at least $100P$ percent of the population using p_U in place of the unknown parameter value p.

Example of Calculating a Binomial Distribution Tolerance Bound

Consider again Problem 2 of the Introduction. Twenty defective units were found in a random sample of 250 units from a production process. An upper (90, 95) tolerance bound N_U on the number of defectives in cartons containing forty-eight units is desired. Thus, N_U is the number of defective units per carton which one can claim with 95 percent confidence will be exceeded by no more than ten percent of the large population of cartons of forty-eight units. One proceeds as follows.

1. As seen above, the upper 95 percent confidence bound on p, based on the given data, is $p_U = 0.115$.
2. An upper 90 percent probability bound on the number of binomial events with $p = 0.115$ and $n = 48$ is desired. The normal distribution approximation to the binomial distribution

will be used to obtain the desired probability interval. The approximating normal distribution has mean $\mu = 48 \times 0.115 = 5.52$ and variance $\sigma^2 = 48 \times 0.115 \times 0.885 = 4.89$.

From equation (16), the 90 percent value of this approximating normal distribution is the solution x to the following.

$$F(x; 0.115, 250) = \Phi[(x + 1/2 - 5.52)/2.21] = 0.90$$

Since $\Phi[1.282] = 0.90$, we obtain

$$(x + 1/2 - 5.52)/2.21 = 1.282$$

and $x = 7.85$. Thus

$$F(8; 0.115, 250) \geq 0.90$$

and

$$F(7; 0.115, 250) < 0.90$$

and from (11) and (12), $N_U = 8$ is the desired upper tolerance bound. That is, one can state with 95 percent confidence that at least 90 percent of the cartons of forty-eight units in the sampled population have no more than eight defective units per carton.

Discussion and Extensions

In the preceding two sections we have described the procedure for obtaining tolerance bounds for Poisson and binomial variables. The same procedure can be applied to obtain tolerance bounds on random variables following other single parameter distributions, such as the exponential distribution, or multiparameter distributions with only one unknown parameter, such as the normal distribution with known standard deviation and the Weibull distribution with known shape parameter.

A tolerance interval should be clearly distinguished from other statistical intervals such as a confidence interval for the unknown population parameter (such as the population mean) and a prediction interval to contain the results of a future sample. Detailed comparisons of these three types of intervals for a normal distribution are provided by Hahn (1970). Hahn and Nelson (1973) give a survey of prediction intervals and their applications.

It needs to be emphasized that the procedures described here and, for that matter all statistical inferences, assume that the given sample is randomly selected from the population of interest. Thus, in the second example it is assumed that the 250 units are a random selection from the same population from which groups of forty-eight units are randomly drawn. This would not be the case, for example, if cartons were packed in the sequence in which units were manufactured and the proportion p of defective units varied over time.

Appendix

Mathematical Development

In this Appendix we outline the mathematical justification for the two step procedure to obtain a tolerance interval for a random variable which follows a known distribution with one unknown parameter. For illustration, we consider the problem of obtaining an upper $100P$ percent tolerance bound on X at a 100γ percent confidence level when X follows a Poisson distribution. Other cases, such as that when X follows a binomial distribution, can be handled similarly.

Let X represent the yearly total number of occurrences of a random event that occurs at a mean rate of λ per year. Assume that X follows a Poisson distribution and the value of λ is not known. Based on the information that during a random period of t years the event has occurred c times, we desire an upper bound N_U on X such that we can claim with a specified degree of confidence that no more than $100(1 - P)$ percent of the distribution of X lies above N_U. In particular, we want to obtain the minimum value N_U such that

$$A = \sum_{x=0}^{N_U} \frac{\lambda^x e^{-\lambda}}{x!} \geq P \qquad \text{(A1)}$$

holds with probability γ or more, irrespective of the value of λ, i.e., we desire the minimum integer N_U that satisfies the following.

$$\Pr\left[\sum_{x=0}^{N_U} \frac{\lambda^x e^{-\lambda}}{x!} \geq P\right] = \gamma \qquad \text{(A2)}$$

The value N_U depends upon the unknown parameter λ. It is easy to show that for a fixed N_U, A decreases monotonically as λ increases. Therefore one needs to take in (A1) the largest value of λ which is consistent with the given sampling information at the 100γ percent confidence level. Thus, it follows that λ in (A2) must equal the value λ_U such that the following is true.

$$\Pr[\lambda \leq \lambda_U] = \gamma$$

Since λ_U is the upper 100γ percent confidence bound on λ, λ_U is the solution for λ to the following.

$$\sum_{x=0}^{c} \frac{(\lambda t)^x e^{-\lambda t}}{x!} = 1 - \gamma$$

We therefore desire the minimum value N_U which satisfies (A1) with $\lambda = \lambda_U$. This is the integer value N_U such that the following inequalities hold.

$$\sum_{x=0}^{N_U-1} \frac{\lambda_U^x e^{-\lambda_U}}{x!} < P$$

$$\sum_{x=0}^{N_U} \frac{\lambda_U^x e^{-\lambda_U}}{x!} \geq P$$

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