TECHNICAL AIDS

by

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Testing Variation Before and After Treatment of a Single Sample

WHEN the variance of a sample of items before the application of a treatment is to be compared with the variance of the same sample after the treatment, the usual F test, which requires two independent samples, cannot be used. Even though this problem was solved over 40 years ago by Pitman (1939) and by Morgan (1939) in papers published consecutively in the same issue of *Biometrika*, the solution is not given in most modern statistics books.

Consider an example in which the variance of a sample of ten items is calculated before a treatment is applied and then again after the treatment is applied. Suppose that the results are as given in Table 1 with X_i equal to the measurement on the ith item before treatment and Y_i equal to the measurement on the ith item after treatment.

The question to be answered is whether the variability has been changed by the treatment. An ordinary F test, here used incorrectly, gives $F = s_X^2/s_Y^2 = 20.95/9.55 = 2.19$ with nine and nine degrees of freedom, which is significant at the 13 percent level (one-sided) or 26 percent level (two-sided). This is to say that the result is not significant at any usual level of significance.

A correlation between the two sets of data may generally be expected because they are based on the same physical items. Consequently the usual F test is not appropriate. The correct procedure takes into account the correlation as follows. Table 1 gives the observed values and those required for calculation, where $x_i = X_i - \bar{X}$ and $y_i = Y_i - \bar{Y}$.

$$\sum x_i^2 = 188.585$$

$$\sum y_i^2 = 85.936$$

$$\sum x_i y_i = 113.000$$

KEY WORDS: Correlated Variances, Correlation, Student's t test

The test statistic is as follows.

$$r = \frac{\sum x_i^2 - \sum y_i^2}{\sqrt{(\sum x_i^2 + \sum y_i^2)^2 - 4(\sum x_i y_i)^2}} = \frac{102.649}{\sqrt{(274.521)^2 - 4(113.000)^2}} = 0.6587$$

If the null hypothesis that $\sigma_X = \sigma_Y$ is true then r is distributed as an ordinary correlation coefficient for a sample of n pairs. Therefore for $n = 10$, the critical value for r is 0.6319 [see for example Owen (1962)] for a two-sided test at the five percent level or for a one-sided test at the two and one-half percent level (with the appropriate algebraic sign). The null hypothesis is rejected here and it is concluded that the variability is lower after the treatment.

Alternatively, as given by Walker and Lev (1953), the same test can be cast in the form of a t test. This is

$$t = \frac{(\sum x_i^2 - \sum y_i^2)\sqrt{n-2}}{2\sqrt{\sum x_i^2 \sum y_i^2 - (\sum x_i y_i)^2}} = \frac{290.3352}{2\sqrt{16206.2406 - 12769.0000}} = 2.4761$$

which is referred to a table of Student's t with $n - 2$ degrees of freedom. For the five percent (two-sided) level of significance with eight degrees of freedom, the critical value is 2.3060. The null hypothesis is rejected just as it was when r was used. The two tests are mathematically equivalent.

It must be emphasized that it is assumed that the variates X and Y follow a bivariate normal distribution.

References

MORGAN, W. A. (1939), "A Test for the Significance of the Difference Between the Two Variances in a Sample from a Normal Bivariate Population," *Biometrika*, Vol. 31, pp. 13–19.

Owen, D. B. (1962), *Handbook of Statistical Tables*, Addison-Wesley Publishing Company, Reading, Massachusetts, p. 510.
Pitman, E. J. G. (1939), "A Note on Normal Correlation," *Biometrika*, Vol. 31, pp. 9-12.
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TABLE 1. Observed Values and those Derived from them for Calculation

Item, i	X_i	Y_i	x_i	y_i	x_i^2	y_i^2	$x_i y_i$
1	18.3	19.3	−6.55	−5.42	42.9025	29.3764	35.501
2	18.4	20.9	−6.45	−3.82	41.6025	14.5924	24.639
3	24.9	23.2	0.05	−1.52	0.0025	2.3104	−0.076
4	22.2	22.8	−2.65	−1.92	7.0225	3.6864	5.088
5	22.5	26.0	−2.35	1.28	5.5225	1.6384	−3.008
6	25.0	25.2	0.15	0.48	0.0225	0.2304	0.072
7	27.1	26.8	2.25	2.08	5.0625	4.3264	4.680
8	29.2	26.2	4.35	1.48	18.9225	2.1904	6.438
9	32.0	27.9	7.15	3.18	51.1225	10.1124	22.737
10	28.9	28.9	4.05	4.18	16.4025	17.4724	16.929
				Totals	188.5850	85.9360	113.000