Taguchi Quality Control

1. Taguchi Quality Control
   - what is quality?
   - the quality loss function
   - expected loss of quality
2. Factor Analysis
   - design of experiments
   - orthogonal design of experiments

MCS 472 Lecture 2
Industrial Math & Computation
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quality as defined by Taguchi

Some obvious points about *what it is not:*

- *product quality $\neq$ product quantity*
  This is a cultural issue. The U.S. is a typical throw-away society with a diverse market, whereas Japan is a homogeneous society. The outcome of world war II promoted mass production and consumption in the U.S., while Japan suffered.
- *quality $\neq$ value*
  Value is subjective and related to the supply-demand-marketing chain. An object (like a worn out teddy bear) can have a tremendous personal value, but no quality.

Definition (Genichi Taguchi, 1986)

An article of good quality performs its intended functions *without variability*, and causes little $\boxed{\text{loss}}$ through *harmful side effects*, including the cost of using it.

the key is *without variability*

Some observations on this definition:

1. The key point is *without variability*.

   Prior to the ideas of Taguchi, people tought the production was okay as long as the products were within the tolerances.

   The reduction of variability is the goal of Taguchi quality control.

2. To enhance the quality we want to reduce loss.

   But we only care about loss caused by variability in the product, not by loss caused by *harmful side effects*.

   Consider for example liquor. The quality of a bottle of liquor is its percentage of alcohol, because its intended effect is intoxication. Harmful side effects are the accidents or fights, etc...

3. Modeling the loss by a quality loss function is central.

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an example

Consider for example the production of shirts, measured by neck size.

Let y be the size of the produced shirt,
i.e.: its neck size; and m the buyer's exact neck size.

We denote by $L(y)$ the loss due to the difference between y and m.

- When $y < m$ (the shirt is too small), return (or discard) the item.
- When $y > m$ (the shirt is too wide), we have to tailor.

Consider the mathematical Taylor expansion of $L(y)$ about m:

$$L(y) = L(m) + \frac{L'(m)}{1!}(y - m) + \frac{L''(m)}{2!}(y - m)^2 + \text{higher order terms.}$$

a quality loss function

$$L(y) = L(m) + \frac{L'(m)}{1!}(y - m) + \frac{L''(m)}{2!}(y - m)^2 + \text{higher order terms.}$$

Why does $L(y)$ model the loss well? Consider:

- $L(m) = 0$, if y equals m, the neck size, then there is no loss.
- $L'(m) = 0$, the loss is minimal for $y = m$.

So, we may represent the loss by

$$L(y) \approx k(y - m)^2, \quad \text{for some constant } k.$$

The constant k is determined by interpolation.

a numerical example

Suppose the critical deviation from m

- for the shirt being too small occurs at $\Delta m^- = 0.5$cm,
- with corresponding cost of rejection being $L^- = \$40$.

Furthermore, the shirt is found too wide

- when it deviates more than one cm from m, i.e., $\Delta m^+ = 1$cm,
- with an associated cost for tailoring set at $L^+ = \$20$.

Since we have different losses when too small or too wide, our quality loss function is piecewise quadratic:

$$L(y) = \begin{cases} k^+(y-m)^2 & \text{for } y \geq m, \\ k^-(y-m)^2 & \text{for } y < m. \end{cases}$$

a numerical quality loss function

Our quality loss function is piecewise quadratic:

$$L(y) = \begin{cases} k^+(y-m)^2 & \text{for } y \geq m, \\ k^-(y-m)^2 & \text{for } y < m. \end{cases}$$

With critical deviations and corresponding costs:

- When a shirt is too small: $\Delta m^- = 0.5$cm, $L^- = \$40$.
- When a shirt is too wide: $\Delta m^+ = 1$cm, $L^+ = \$20$.

We determine k^+ as follows ($\Delta m^+ = y - m, y \geq m$):

$$L^+ = k^+(\Delta m^+)^2 \Rightarrow 20 = k^+(1.0)^2 \Rightarrow k^+ = 20.$$

Similarly, k^- is computed as follows ($\Delta m^- = m - y, y < m$):

$$L^- = k^-(\Delta m^-)^2 \Rightarrow 40 = k^-(0.5)^2 \Rightarrow k^- = 160.$$

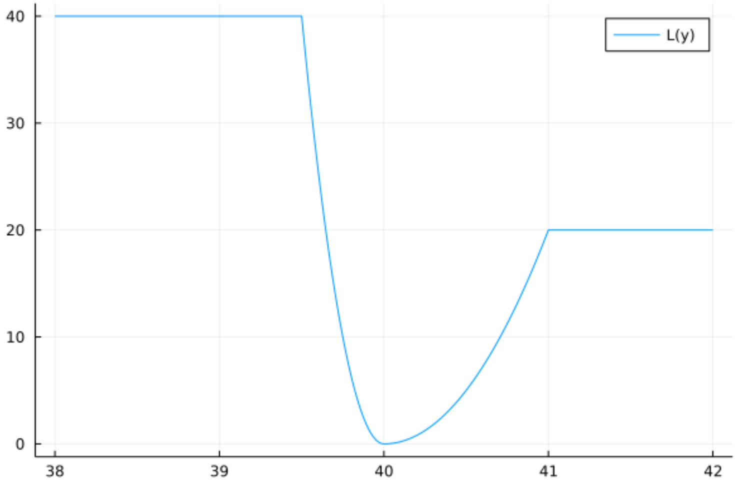
the complete piecewise quality loss function

Let $m = 40$cm be the neck size of the buyer.

$$L(y) = \begin{cases} 20 & \text{for } y \geq 41 \\ 20(y-40)^2 & \text{for } 40 \leq y < 41 \\ 160(y-40)^2 & \text{for } 39.5 < y < 41 \\ 40 & \text{for } y \leq 39.5 \end{cases}$$

```
function loss(y)
    if y >= 41                     # $20 cost for too wide
        return 20
    elseif y >= 40
        return 20*(y-40)^2
    elseif y > 39.5
        return 160*(y-40)^2
    else                           # $40 cost for too small
        return 40
    end
end
```

the plot of the quality loss function $L(y)$



Missing the target causes loss.

definition of the quality loss function

Definition (quality loss function)

Denote by X the random variable modeling the outcome of production.

Let θ denote the target value for the production.

Then the *quality loss function* is

$$L(X, \theta) = k(X - \theta)^2, \quad \text{for some constant } k.$$

We call the constant k the *loss coefficient*.

The cumulative probability distribution function is denoted by $F(x)$,

- with mean $\mu = E[X]$ and
- standard deviation $\sigma^2 = E[(X - \mu)^2]$.

Given μ and σ, what is the expected loss?

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expected loss of quality

Given a quality loss function L, the expected loss $E[L(X,\theta)]$

$$
\begin{aligned}
&= \int_{-\infty}^{+\infty} k(x-\theta)^2 dF(x) \ = \int_{-\infty}^{+\infty} k(x-\mu+\mu-\theta)^2 dF(x) \\
&= \int_{-\infty}^{+\infty} k[(x-\mu)+(\mu-\theta)]^2 dF(x) \\
&= \int_{-\infty}^{+\infty} k(x-\mu)^2 dF(x) + \int_{-\infty}^{+\infty} 2k(x-\mu)(\mu-\theta) dF(x) \\
&\qquad + \int_{-\infty}^{+\infty} k(\mu-\theta)^2 dF(x) \\
&= k\int_{-\infty}^{+\infty} (x-\mu)^2 dF(x) + 2k(\mu-\theta)\left(\int_{-\infty}^{+\infty} x dF(x) - \mu \int_{-\infty}^{+\infty} dF(x)\right) \\
&\quad + k(\mu-\theta)^2 \int_{-\infty}^{+\infty} dF(x)
\end{aligned}
$$

the terms in the expected loss

$$k \int_{-\infty}^{+\infty} (x-\mu)^2 dF(x) + 2k(\mu-\theta) \left(\int_{-\infty}^{+\infty} x dF(x) - \mu \int_{-\infty}^{+\infty} dF(x) \right)$$

$$+ k(\mu-\theta)^2 \int_{-\infty}^{+\infty} dF(x) \quad \text{is simplified with}$$

$$\sigma^2 = \int_{-\infty}^{+\infty} (x-\mu)^2 dF(x), \ \mu = \int_{-\infty}^{+\infty} x dF(x), \text{ and } \int_{-\infty}^{+\infty} dF(x) = 1:$$

$$\begin{array}{rcl} E[L(X,\theta)] & = & k\sigma^2 + 2k(\mu-\theta)(\mu-\mu) + k(\mu-\theta)^2 \\ & = & k\sigma^2 + k(\mu-\theta)^2 \end{array}$$

The expected loss consists of two terms:

1. the loss due to variability $k\sigma^2$ and
2. $k(\mu-\theta)^2$ is the loss due to missing the target.

expected loss of quality

Definition (Expected Loss)

Let $L(X) = k(X - \theta)^2$ be the quality loss function, where

- k is the loss coefficient,
- θ is the target value of the production,
- X is a random variable which models the production, with
  - $\mu = E[X]$, the mean value of the production,
  - $\sigma^2 = E[(X - \mu)^2]$, the standard deviation.

The *expected loss* is $E[L(X, \theta)] = k\sigma^2 + k(\mu - \theta)^2$.

an example on the cost of variance

We consider the color density of televisions.

The loss coefficient k is \$1.25, so $L(y) = 1.25(y - m)^2$,
where y is the produced color density, m is the target color density.
The expected loss is $1.25(\sigma^2 + (\mu - m)^2)$.

For televisions produced in San Diego and Tokyo, $\mu - m = 0$, but

- $\sigma^2 = 8.33$ for the San Diego plant, and
- $\sigma^2 = 2.78$ for the Tokyo plant.

The expected loss per television is then computed as

- $1.25(8.33 + 0) = \$10.41$ per unit for the San Diego plant, and
- $1.25(2.78 + 0) = \$3.48$ per unit for the Tokyo plant.

an exercise

Exercise 1:
Suppose an item costs \$117 to manufacture.

If the item misses the target of 12 by a margin of 3,
then the item must be discarded.

1. Determine the quality loss function.

The production has a mean of 11.7 and standard deviation of 1.0.

2. Suppose an effort of \$20 per item reduces the standard deviation to 0.95. Is this a worthwhile effort?

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factor analysis

Which factors have the most effect on quality?

Examining a production line, we could do better

- if only we had better machines,
- if only we had better trained people,
- if only we had more production time,
- ...

Of the factors listed above, which is most important?

How do we relate the factors to quality?

loss of quality

Suppose we have three factors x_1, x_2, x_3.

$L(x_1, x_2, x_3)$ is the loss of quality as a function of the three factors.

Apply Taylor expansion again and use a linear approximation:

$$L(x_1, x_2, x_3) = \alpha_0 + \alpha_1 x_1 + \alpha_2 x_2 + \alpha_3 x_3.$$

The factor with the largest coefficient has the largest impact.

How do we determine the coefficients α_1, α_2, α_3?

experimental factorial design

To determine the coefficients in

$$L(x_1, x_2, x_3) = \alpha_0 + \alpha_1 x_1 + \alpha_2 x_2 + \alpha_3 x_3$$

we can run the production many times, for

$$-1 \leq x_i \leq +1, \quad i = 1, 2, 3,$$

setting each factor once low (-1) and once high $(+1)$.

For example:

$$\begin{array}{rcl} -L(-1, x_2 = c_2, x_3 = c_3) & = & \alpha_0 - \alpha_1 + \alpha_2 c_2 + \alpha_3 c_3 \\ +L(+1, x_2 = c_2, x_3 = c_3) & = & \alpha_0 + \alpha_1 + \alpha_2 c_2 + \alpha_3 c_3 \\ \hline L(+1, x_2 = c_2, x_3 = c_3) & = & 2\alpha_1 \end{array}$$

Problem: $2^3 = 8$ *runs needed.*

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an orthogonal array of experiments

We run four experiments and observe L_1, L_2, L_3, L_4:

	x_1	x_2	x_3	
1:	$+1$	-1	-1	L_1
2:	-1	$+1$	-1	L_2
3:	$+1$	$+1$	$+1$	L_3
4:	-1	-1	$+1$	L_4

Observe:

- Every column sums up to zero.
- The inner product of every column with every other column is zero.

The columns are orthogonal to each other.

apply linear algebra

Apply the array to $L(x_1, x_2, x_3) = \alpha_0 + \alpha_1 x_1 + \alpha_2 x_2 + \alpha_3 x_3$.

$$
\begin{array}{rcrcl}
L(+1,-1,-1) & = & \alpha_0 + \alpha_1 - \alpha_2 - \alpha_3 & = & L_1 \\
L(-1,+1,-1) & = & \alpha_0 - \alpha_1 + \alpha_2 - \alpha_3 & = & L_2 \\
L(+1,-1,+1) & = & \alpha_0 + \alpha_1 - \alpha_2 + \alpha_3 & = & L_3 \\
L(-1,+1,+1) & = & \alpha_0 - \alpha_1 + \alpha_2 + \alpha_3 & = & L_4 \\
\hline
 & & 4\alpha_0 & = & L_1 + L_2 + L_3 + L_4
\end{array}
$$

In matrix-vector notation:

$$
\begin{bmatrix} +1 & -1 & -1 \\ -1 & +1 & -1 \\ +1 & -1 & +1 \\ -1 & +1 & +1 \end{bmatrix}
\begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{bmatrix}
=
\begin{bmatrix} L_1 - \alpha_0 \\ L_2 - \alpha_0 \\ L_3 - \alpha_0 \\ L_4 - \alpha_0 \end{bmatrix}.
$$

an orthogonal matrix

In matrix-vector notation, $A\boldsymbol{\alpha} = \mathbf{b}$:

$$\begin{bmatrix} +1 & -1 & -1 \\ -1 & +1 & -1 \\ +1 & -1 & +1 \\ -1 & +1 & +1 \end{bmatrix} \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{bmatrix} = \begin{bmatrix} L_1 - \alpha_0 \\ L_2 - \alpha_0 \\ L_3 - \alpha_0 \\ L_4 - \alpha_0 \end{bmatrix}.$$

By the orthogonality $A^T A = 4I$, where I is the identity matrix.

So we simply read off the values for $\boldsymbol{\alpha}$.

a second exercise

Exercise 2:

Construct a 9-by-4 orthogonal array
using an equal amount of -1, 0, and $+1$ per column.

- the sum of the element in each column is zero, and
- the columns are orthogonal to each other.

Explain your work.

bibliography

- G. Taguchi. *Introduction to Quality Engineering. Designing Quality into Products and Processes.* Asian Productivity Organization, 1986.
- *Handbook of Total Quality Management*, edited by Christian N. Madu. Springer-Verlag, 1998.