On the Use of Tolerance Intervals in Acceptance Sampling by Attributes

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A type of specification used in acceptance sampling of a lot of items involves the use of tolerance intervals. The producer is required to make the following type of statement about a lot, "With 100γ% confidence, at least 100P% of the items in the lot are acceptable, i.e., within specifications." In this situation, the producer has freedom in selecting his sample size; all he must do is perform sufficient sampling to permit his making the required tolerance interval statement. However, in exercising this freedom, he must use care to properly assess the risk to the consumer if he attempts to meet the specification with a multiple type sampling plan. This paper evaluates this risk when the inspection is on a certain attributes basis.

Introduction

CONSIDER the situation in which a producer inspects a lot of items on an attributes basis. He is required to make a tolerance interval statement of the following form: "With 100γ% confidence, at least 100P% of the items in the lot are acceptable, i.e., within specifications." Henceforth, an out-of-specification item is labeled a defective.

With a fixed sample size and with γ and P specified, a minimum amount of sampling is required to permit making this statement even with no defectives in the sample. In practice, if the producer finds one defective in this initial sample, he might be tempted to sample an additional number of items, hoping to find no further defectives, and thereby supposedly justify permitting him to make the required tolerance statement on the basis of just one defective in his entire sample. As long as he continues to have hopes of eventually being able to make the required tolerance statement, he may continue this approach rather than face the prospect of performing 100% inspection of the entire lot.

Clearly, there is no objection to the sequential sampling approach per se. However, the producer must recognize that with this approach, the risk to the consumer will not be what it may appear to be on the surface. This is illustrated for a commonly encountered specification with $P = .95$ and $\gamma = .95$ (a 95:95 tolerance limit specification). Suggestions for altering the sample sizes to make the sequential or multiple sampling approach valid are then offered.

Fixed Sample Size

For attributes inspection, a tolerance interval statement of the form, "With 100γ% confidence, at least 100P% of the items in the lot are within specifications," is equivalent to requiring that the one-sided upper 100γ% confidence limit on the lot proportion defective, p, be $(1 - P)$, where p is the probability that an item is defective. Assuming that the lot size is large relative to the sample size such that it may be considered infinite, p may then be regarded as the binomial distribution parameter.

If the sample size were fixed in the sense that the lot would either be accepted or rejected on the basis of the single sample result, then the procedure would be to select a random sample of n items from the lot and count the number of defectives, k. The 100γ% upper confidence limit on the parameter p is the largest value of p such that

$$\sum_{i=0}^{k} \binom{n}{i} p^i (1 - p)^{n-i} \leq (1 - \gamma).$$

For P and γ specified, the minimum sample size can be found as a function of k using this equation. For $\gamma = .95$ and $P = .95$, which are commonly

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TABLE 1. Minimum Sample Sizes for γ = .95, P = .95

k	Minimum Sample Size
0	59
1	93
2	124
3	153
4	181

specified, Table 1 gives this minimum sample size as a function of k, labeled the acceptance number.

To illustrate with a fixed sample size plan in which 124 items are selected, if there are two or fewer defectives in the sample, the lot is accepted, i.e., the required tolerance limit statement can be made.

Truncated Multiple Sampling

Suppose that an initial minimum size sample of 59 is drawn on the basis of Table 1. Further suppose that one or more defectives are found in this sample such that the required tolerance statement cannot be made. The producer might then decide to extend the original sample in the hopes of meeting the specification on the basis of further sample results. For example, he may find one defective in the original sample, draw an additional 34 items observing no defectives, and then assert that the specification has been met on the basis of the one defective found in the total of 93 items (see Table 1). Continuing in this approach, suppose that the producer decides to continue sampling, always using the minimum number of additional observations as dictated by Table 1, until he either accepts the lot or finds ≤ 5 defectives, at which point he performs 100% inspection on the remaining items. It will be shown that such an approach increases the probability of acceptance above .05 when $p = .05$ and thus does not justify the $\gamma = .95, P = .95$ statement.

There are 16 possible "Routes" by which the producer might assert that he has met the specification under the above procedure. These are listed in Table 2. For example, with Route 6, the producer finds one defective in the initial sample of size 59, so he samples an additional 34 items. He finds two defectives in this sample and, recognizing from Table 1 that he cannot meet the specification now without a total sample of size 153, he samples an additional 60 items. Finding no defectives, he might assert that he has met the specification, having found three items in the 153 sampled. This would be a legitimate assertion had he started out fully intending to inspect the full 153 items, but it does not give the proper assurance when done sequentially, as will be seen in the following section.

TABLE 2. Acceptance Routes Used by the Producer (The tabular entry is the number of defectives found)

Total Sample Size	59	93	124	153	181
Acceptance Numbers	0	1	2	3	4
Route Number					
1	0				
2	1	0			
3	1	1	0		
4	2	—	0		
5	1	1	1	0	
6	1	2	—	0	
7	2	—	1	0	
8	3	—	—	0	
9	1	1	1	1	0
10	1	1	2	—	0
11	1	2	—	1	0
12	1	3	—	—	0
13	2	—	1	1	0
14	2	—	2	—	0
15	3	—	—	1	0
16	4	—	—	—	0

Probability of Accepting the Lot Versus p

The probability of accepting the lot can be found as a function of $100p$, the true percent defective, for the producer's plan as given in Table 2. To find this, the probability associated with each acceptance route listed in Table 2 must be calculated, and the results appropriately summed.

The calculation is illustrated for routes 5–8, all of which result in a total of 153 observations.

Route 5:

$$(59)p(1-p)^{58}(34)p(1-p)^{33}(31)p(1-p)^{30} \cdot (1-p)^{29} = 62{,}186\ p^3(1-p)^{150}$$

Route 6:

$$(59)p(1-p)^{58}[(34)(33)/2]p^2(1-p)^{32} \cdot (1-p)^{60} = 33{,}099\ p^3(1-p)^{150}$$

Route 7:

$$[(59)(58)/2]p^2(1-p)^{57}(65)p(1-p)^{64} \cdot (1-p)^{29} = 111{,}215\ p^3(1-p)^{150}$$

Route 8:

$$[(59)(58)(57)/6]p^3(1-p)^{56} \cdot (1-p)^{94} = 32{,}509\ p^3(1-p)^{150}$$

$$\text{TOTAL} = 239{,}009\ p^3(1-p)^{150}$$

TABLE 3. Constants for Computing the Probability, θ, That the Sample Size Will Be Exactly n where $\theta = Cp^a(1-p)^b$

n	C	a	b
59	1	0	59
93	59	1	92
124	3,717	2	122
153	239,009	3	150
181	15,370,267	4	177

This is the probability that the lot will be accepted with a sample size of *exactly* 153 items as a function of p, where $(100p)$ % is the lot percent defective. If the lot percent defective were 2%, say, this probability is .09235. The formulas giving the probability of accepting the lot with this plan as a function of total sample size are given in Table 3. The probabilities for $p = .05$ are given in Table 4.

Table 4 points out the flaw in the acceptance plan used by the producer. Say that a lot with exactly 5% defectives is submitted for inspection. With the producer's plan as described, there is a probability of 0.11719 that this will be accepted with a sample of size 181 or less. However, the *intent* of the specification is that such a lot be accepted with a probability not to exceed 0.05. Clearly, although a plan with four or fewer defectives out of a total *fixed* sample size of 181 meets this criterion, this sequential plan used by the producer which *may eventually result* in the same sample outcome does *not* meet the criterion; the protection to the consumer is much less than intended.

Valid Multiple Sampling Approaches

It was indicated earlier that a multiple acceptance sampling approach poses no *inherent* difficulties in meeting this type of specification. The problem encountered with the approach thus far discussed was not the multiple sampling feature, but rather, the lack of sufficiently large sample sizes at the various stages in the sampling process.

Before indicating how the sample sizes of Table 1 should be increased, it should be emphasized that any sampling plan—single, double, multiple or pure sequential, for which the probability is .05 of accepting a lot containing 5% defectives—is a valid plan from point of view of meeting the 95:95 tolerance limit criterion. One readily available potential source of such plans is MIL-STD-105D [3]. A discussion of specific MIL-STD-105D plans which meets the 95:95 tolerance limit criterion is deferred for the present.

Consider a family of plans which retain the general features of the plan schematically represented in Table 2 except for changing the sample sizes at each stage in the sequential process. Specifically, replace the incremental sample sizes of 59, 34, 31, 29 and 28 in Table 2 by N_1, N_2, $\cdots N_5$. Then, Table 5 gives some specific solutions for the N_i where this family of plans is constructed based on the parameter:

$p_{i;.05}$ = probability that a lot containing 5% defectives is accepted with a *total* sample of size $\sum_{j=1}^{i} N_j;\ i = 1, 2 \cdots$

The family of plans under consideration here fixes the ratio $r = p_{i+1;.05}/p_{i;.05}$ such that, conceptually, one could allow for the possibility of continuing the sampling indefinitely beyond N_5 while still retaining the feature that the probability of accepting a lot containing 5% defectives is $\leq .05$. It is easy to see that $r = 1 - 20p_{1;.05}$ is the value for r which provides this protection. Dropping the subscripts on $p_{1;.05}$:

$$\begin{aligned} .05 &= p + rp + r^2p + \cdots \\ &= p(1 + r + r^2 + \cdots) \\ &= p/(1-r) \text{ from which } r = 1 - 20p \end{aligned}$$

TABLE 4. Probability of Accepting the Lot for p = .05

Cumulative Sample Size	Acceptance Number	$\Sigma\theta_i$ = Probability of Accepting Lot
59	0	.04849
93	1	.07482
124	2	.09262
153	3	.10623
181	4	.11719

TABLE 5. Sample Sizes, N_i, For Five Sampling Plans with $\gamma = .95$; p = .95

$p_{1;.05}$ =	.01	.02	.03	.04	.05
r =	.8	.6	.4	.2	0
N_1	90	77	69	63	59
N_2	35	38	43	55	—
N_3	33	37	46	61	—
N_4	32	38	48	65	—
N_5	32	39	50	68	—

$p_{1;.05}$ = probability of accepting the lot with the initial sample when there are 5% defectives in the lot.
$r = p_{2;.05}/p_{1;.05} = p_{3;.05}/p_{2;.05}$, etc.
where $p_{i;.05}$ = probability of accepting this lot with a total sample size of $\Sigma_{j=1}^{i} N_j$.

TABLE 6. Average Total Inspection with 95:95 Plan for Lot Containing N Items

Plan with $p_{1;.05}$ =	.01	.02	.03	.04	.05*
% Defective					
.1	93.2	80.0	72.1	66.7 + .0001N	55.6 + .0573N
.2	96.7 + .0002N	83.3 + .0001N	75.0	70.9 + .0002N	52.4 + .1114N
.5	107.5 + .0042N	93.8 + .0040N	82.6 + .0053N	84.7 + .0093N	43.9 + .2560N
1	118.5 + .0576N	105.4 + .0564N	86.2 + .0689N	95.2 + .1003N	32.6 + .4473N
2	89.2 + .3876N	81.3 + .3761N	58.9 + .4107N	57.9 + .4837N	17.9 + .6964N
3	41.4 + .7249N	39.0 + .7051N	31.1 + .7240N	23.4 + .7628N	9.8 + .8342N
4	14.6 + .9009N	14.7 + .8833N	10.7 + .8847N	9.1 + .8935N	5.3 + .9100N

* When $p_{1;.05}$ = .05, this is a single sample plan

To illustrate, if one were to follow the acceptance routes given schematically (Table 5) in Table 2 but use the incremental sample sizes 69, 43, 46, 48, 50 (Column 3 of Table 5), then the probability of accepting a lot containing 5% defectives at stage 1 (N_1 = 69) is $\leq$ .03; at stage 2, it is $\leq$ (.03)(.4) = .012; etc. (The $\leq$ signs apply because the N_i must necessarily be integers, and in solving for the N_i, the solutions were rounded up.)

Choice of Specific Plan

Within this family of plans, one has a choice of which to use. This decision depends on the expected quality of the lot submitted for inspection and the lot size and is logically based on minimizing the average total inspection (ATI). In calculating ATI, the assumptions are made that a decision will be reached at least by the time the fifth subsample is drawn, and that when a lot is rejected it will be 100% inspected. (Thus, it does not matter at what point in the inspection process a lot is rejected; 100% of the items will be inspected in this event.)

Average total inspection as a function of the percent defective in a lot and lot size, N, is given in Table 6. The coefficient of N is the probability that the lot will be rejected and therefore may be used to construct the OC curves.

For example, if one were dealing with a lot size of N = 1000, and anticipated a 0.5% rate of defectives, then the average total inspecton for the various plans is given in Table 7. In this situation, (without bothering to interpolate), he would select the plan corresponding to $p_{1;.05}$ = .03 where the N_i are 60, 43, 46, 48, 50 from Table 5. Note the large advantage of this plan over the single sample plan corresponding to $p_{1;.05}$ = .05.

TABLE 7. Average Total Inspection with 95:95 Plan for 0.5% Defectives and Lot Size of 1000

Plan with $p_{1;.05}$	ATI
.01	111.7
.02	97.8
.03	87.9
.04	94.0
.05	299.9

MIL-STD-105D Multiple Sampling Plans

It was mentioned earlier that readily available multiple plans are found in MIL-STD-105D [3]. In using this reference to construct a plan, one would simply search for OC curves which pass through the point (.05, .05). In doing this, the plan corresponding to sample letter K, AQL = 0.65, meets the criterion. Another possibility which comes close is sample letter L, AQL = 1.0, although it does not quite satisfy the 95:95 tolerance limit criterion. Consider how the plan K, AQL = 0.65, compares with those just discussed. This plan calls for constant N_i at each inspection step, with N_i = 32 for all steps. A final decision is made to accept or reject when the total sample size is $\leq$ 224.

In the context of the current discussion, the MIL-STD plan is very similar to the family of plans presented in the previous section. To see this, the plan corresponding to sample letter K, AQL = 0.65, is given in Table 8. The last two columns of Table 8 present plan K in the same context as before. Recall that in the situation under discussion, the decision is made to either accept or 100% inspect upon rejection. It makes no difference at which point in the inspection process the decision is made to reject the lot; 100% inspection will be required whenever this decision is reached. Clearly, Plan K will tend to reject a lot of poor quality sooner which, in other contexts, would be an advantage.

Plan K is evaluated like the others except that in view of the possibility of earlier rejection, acceptance routes 8, 10, 12, 14, 15 and 16 in Table 2 are not included. Keeping this in mind, the average total inspection for this plan is given in Table 9 to com-

TABLE 8. Multiple Sampling Plan for Sample Letter K, AQL = .065, from MIL-STD-105

Cumulative Sample Size	Ac	Re	N_i	Ac
32	#	2	64	0
64	0	3	64	1
96	0	3	32	2
128	1	4	32	3
160	2	4	32	4
192	3	5		
224	4	5		

Acceptance not permitted at this sample size

TABLE 9. Average Total Inspection for MIL-STD-105D Multiple Sampling Plan K

% Defective	ATI
.1	68.2
.2	72.4 + .0004N
.5	86.2 + .0013N
1	94.2 + .0820N
2	69.5 + 4114N
3	33.0 + 7153N
4	12.8 + .8780N

pare directly with the results in Table 6. Just how this average total inspection compares with those given in Table 6 depends on both the lot percent defective and the lot size. It is not the intent of this paper to compare the plans in any great detail, but the results given in Tables 6 and 9 can be used to make the comparisons in specific instances.

Summary

The principal aim of this presentation is to emphasize that one *cannot* validly extend one's sampling if the acceptance criterion is not met with the initial single sample when the initial sample size is based on a fixed sample size plan. Attention is then directed to multiple sampling plans. One satisfactory, easily available source of such plans is MIL-STD-105D [3]. Some consideration is given to developing a family of plans which, conceptually, allows for the sequential sampling to continue indefinitely while still keeping the probability of accepting a lot containing 5% defectives below .05. In practice, the advantages of this family of plans over the MIL-STD multiple sampling plans appear to be small, although a complete characterization of the plans was not made. In circumstances other than those discussed in this application, (i.e., 100% inspection upon rejection of the lot) the MIL-STD plans would have the advantage in that they lead to earlier rejection of unacceptable quality material. The MIL-STD plans are also simple to administer and are readily available.

For a further discussion of multiple attribute sampling plans, the reader is referred to the books by Dodge and Romig [1] and by the Statistical Research Group of Columbia University [2].

APPENDIX

The specific sampling plans presented used values for $p_{1;.05}$ equal to .01, .02, .03, and .04. For other values of $p_{1;.05}$, the following formulas may be used to compute the N_i (rounding up to the nearest integer in each instance).

$$
\begin{aligned}
N_1 &= -19.5 \ln p_{1;.05} \\
N_{i+1} &= -19.5 \ln (C_i r/.05 C_{i+1}) + 1, \; i = 1, \cdots, 4 \\
r &= 1 - 20 p_{1;.05} \\
C_1 &= 1 \\
C_2 &= N_1 \\
C_3 &= N_1 N_2 + N_1(N_1 - 1)/2 \\
C_4 &= N_1 N_2 N_3 + N_1 N_2 (N_2 - 1)/2 \\
&\quad + N_1(N_1 - 1)(N_2 + N_3)/2 \\
&\quad + N_1(N_1 - 1)(N_1 - 2)/6 \\
C_5 &= N_1 N_2 N_3 N_4 + N_1 N_2 N_3 (N_3 - 1)/2 \\
&\quad + N_1 N_2 (N_2 - 1)(N_3 + N_4)/2 \\
&\quad + N_1 N_2 (N_2 - 1)(N_2 - 2)/6 \\
&\quad + N_1(N_1 - 1)(N_2 + N_3) N_4/2 \\
&\quad + N_1(N_1 - 1)(N_2 + N_3)(N_2 + N_3 - 1)/4 \\
&\quad + N_1(N_1 - 1)(N_1 - 2)(N_2 + N_3 \\
&\quad + N_4)/6 \\
&\quad + N_1(N_1 - 1)(N_1 - 2)(N_1 - 3)/24.
\end{aligned}
$$

In the discussion, the N_i were not computed beyond N_5, but it was mentioned that sampling could conceptually proceed indefinitely beyond this point. The calculation of the N_i values beyond N_5 becomes burdensome because the number of "acceptance routes" gets so large. However, the N_6 values were computed, and at least some extrapolation beyond N_6 is rather apparent. The N_6 values which can form an additional row in Table 5 are 32, 40, 51 and 70, respectively.

References

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